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A Semiparametric Latent Trait Model for Response Times in Tests

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Abstract

In this paper, we propose a latent trait model for the response times in the items of a psychological test. The latent trait model consists of two components. The first component is a model for the marginal response time distributions in the single items. It is based on an approximation of the marginal response time distributions via a spline hazard model. The second component is a factor copula model. It relates the marginal response time distributions to a latent trait that represents the work pace of the respondents. The factor copula model is based on a normal mixture copula. The combination of the spline hazard model with the factor copula model results in a response time model of high flexibility. It subsumes or is able to approximate several well-known models for response times in tests. It can be used as a measurement model in order to estimate the work pace underlying the response times of a respondent. The model can be fitted to data with a two-step maximum likelihood estimator. The performance of the estimator is investigated in a simulation study. We also demonstrate the model's applicability to a real data set.

Keywords: Latent Trait Model, Response Time Model, Spline Hazard Model, Factor Copula Model

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Psychological tests are often computer assisted. In this case, one usually records not only the responses, but also the response times needed to give the responses (e.g., Shin, Kerzabi, Joo, Robin, & Yamamoto, 2020). The response times provide additional information about the response process that can be used for the detection of irregular response modes (e.g., Ranger & Kuhn, 2017), the selection of items in computer adaptive testing (e.g., Diedenhofen & Musch, 2018) or as collateral information about the respondents' characteristics (e.g., Hohensinn & Kubinger, 2017; see Kyllonen & Zu (2016) and Lee & Chen (2011) for an overview). Sometimes (e.g., in objective tests of personality), the response times are even more important for psychological assessment than the responses (Ortner, Proyer, & Kubinger, 2006).

Some of the applications of response times in psychological assessment require a latent trait model that relates the response times to the latent work pace of the respondents. In contrast to the responses for which standard item response models are established (Embretson & Reise, 2000), there is still no consensus on the latent trait model to be used for the response times (Kyllonen & Zu, 2016). Two classes of models can be distinguished, the class of parametric and the class of semiparametric latent trait models. Parametric latent trait models embed a standard response time distribution into a latent trait framework, either by relating the work pace to the possibly transformed moments of the response times or to the parameters of the response time distribution (Molenaar, Tuerlinckx, & van der Maas, 2015; Rouder, Sun, Speckman, Lu, & Zhou, 2003). Such models are, for example, based on the exponential distribution (Scheiblechner, 1979), the Weibull distribution (Roskam, 1997), the Gamma distribution (Maris, 1993) or the log-normal distribution (van der Linden, 2006). Using a specific distribution is a strong distributional assumption that might be wrong in practice. In case the assumed distribution is not capable to represent the response time distribution well, any type of inference about the respondents might be wrong. Strong distributional assumptions are avoided by semiparametric response time models.

Semiparametric response time models are models with a parametric and a nonparametric component (van der Vaart, 1995). The linear transformation model, for example, claims that a transformation of the response times can be decomposed into a linear function of the work pace and an additional residual term. The residual term is assumed to be distributed according to a standard distribution, usually the normal distribution or the Gumbel distribution. No strong assumptions, however, are made about the transformation besides its monotony. The linear transformation model has been implemented for response times in tests by Wang, Chang, and Douglas (2013), Wang, Fan, Chang, and Douglas (2013), Ranger and Ortner (2013) and Ranger and Kuhn (2013). Although the linear transformation model is rather general and subsumes well known models like the log-normal factor model (van der Linden, 2006) or the proportional hazards model with random effects (Ranger & Ortner, 2013), there is still some inflexibility as an assumption about the distribution of the residuals has to be made. This can be avoided by modeling the distribution of the residuals in addition to the unknown transformation. Such a generalization was proposed by Ranger and Kuhn (2012) for discrete response times and by Ranger and Kuhn (2015) for continuous response times.

The linear transformation model is more flexible than a parametric response time model. It is, however, based on the strong assumption that a monotone transformation is capable to transform the response times in such a way that the they can be modeled by a standard linear latent trait model. This implies that the transformed response times are linearly related to the latent work pace, are variance homogeneous and have a specific distribution like, for example, a normal one. Such a transformation might not exist in real data. We will illustrate this point with the help of response times that were distributed similarly to the response times in a real data set; see the last section of the paper. The response times were generated with the response time model proposed in this paper, but were analyzed with the linear transformation model of Klein Entink, van der Linden, and Fox (2011). In the model of Klein Entink et al. (2011), the response times are first transformed according to a Box-Cox transformation and then analyzed with a standard factor model. The transformed response times were approximately normally distributed. This is illustrated by the Q-Q-plot of the transformed response times in an exemplary item given in Figure 1, Plot 1; note that the sample quantiles are in good agreement to the theoretical quantiles of the normal distribution. The conditional expectation of the transformed response times was also an almost linear function of the true latent work pace; see Figure 1, Plot 2 where the black curve represents the expectation of the transformed response time in the item conditional on the latent work pace θ. The black curve is very close to the red line that provides the best linear approximation. The transformed response times in the item, however, were not variance homogeneous. This is illustrated in Figure 1, Plot 3, where the conditional variance of the transformed response times is plotted against the latent work pace θ. The conditional variance increases when the latent work pace deviates from zero. This impairs the validity of the trait inference. Figure 1, Plot 4 depicts the coverage frequency of confidence intervals ($\alpha = 0.05$) for the latent work pace. The confidence intervals were determined with the transformed response times in all items and the linear factor model. The coverage frequencies deviate strongly from the intended coverage probability of 0.95 (red line) for values near -2 and 2.

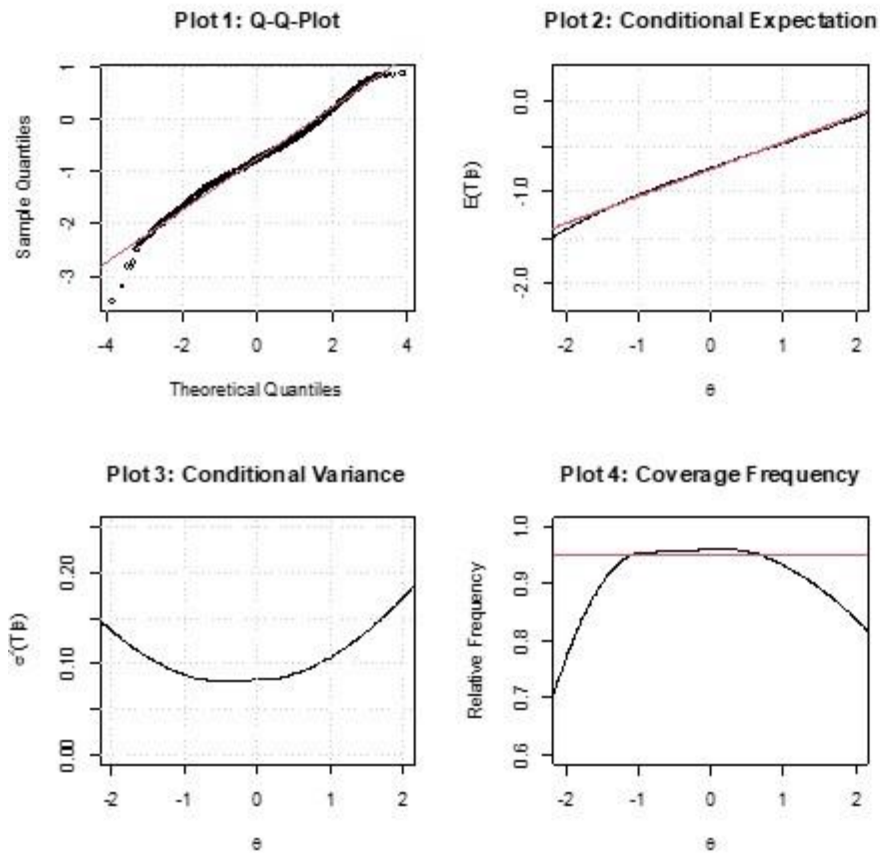

Figure 1

Validity of the Linear Transformation Model for Response Times in a Test

Note. Plot 1 shows a Q-Q-plot of the transformed response times in an exemplary item of the test. A deviation of the black curve from the red line indicates deviations from the normal distribution. Plot 2 visualizes the relation of the conditional expectation $E(T|\theta)$ of the transformed response time to the level of the underlying latent work pace θ in the item (black line) as well as the best linear approximation (red line). Plot 3 visualizes the relation of the conditional variance $\sigma^2(T|\theta)$ of the transformed response time to the level of the underlying latent work pace θ in the item. Plot 4 gives the coverage frequencies of confidence intervals for the work pace (black line) that were determined with the transformed response times in all items and a linear factor model. The red line denotes the intended coverage frequency of 0.95.

The linear transformation model makes the strong assumption that after the transformation, the transformed response times are related to the latent work pace according to a standard linear trait model. In this paper, we extend the linear transformation model by modeling the association of the transformed response times and the latent work pace in a more flexible way. The core of the model is the factor copula model of Krupskii and Joe (2013). According to the factor copula model, the transformed response times are related to the latent work pace via a copula. Thereby, the distribution of the response times in the single items and their relation to the work pace can be modeled separately. The distribution of the response times is modeled with a spline hazard model. In the spline hazard model, the hazard function of the response time distribution is represented via M-splines (Ramsay, 1988). The association between the response times and the latent work pace is modeled with a normal mixture copula (Nikoloulopoulos & Karlis, 2009; Tewari, Giering, & Raghunathan, 2011). The normal mixture copula is generated by a mixture of bivariate normal distributions. The resulting response time model has several advantages. The model is more flexible than previous response time models and subsumes several well-known response time models as special cases. Its building blocks, the hazard function and the normal mixture copula, can give insight into the response process. Furthermore, the model can be used as a measurement model in order to estimate the work pace of the respondents.

The paper is organized as follows. First, we describe the main idea behind the factor copula model. Then, we describe how this approach can be implemented in the present context. Model fitting and approaches to model inference will be addressed next, as well as the usage of the model as a measurement model. Finally, the model is applied to empirical data.

The Factor Copula Model

The core of a (one-dimensional) latent trait model is the specification of how a latent trait is related to its observable indicators. The standard factor model, for example, states that each indicator can be decomposed into a linear combination of the trait and an additional residual term. Here, we model the relation of the work pace to the response times with the factor copula model proposed by Krupskii and Joe (2013) and Nikoloulopoulos and Joe (2015).

Denote by θ the latent work pace of a respondent and let T_g be his/her response time in item g. In the population of the potential respondents, the latent trait and the response time are distributed according to a bivariate distribution with cumulative distribution function $\mathrm{F}_{T_g,\theta}\left(t_g,\theta\right)$ and marginal cumulative distribution functions $\mathrm{F}_{T_g}\left(t_g\right)$ and $\mathrm{F}_{\theta}\left(\theta\right)$. According to Sklar's theorem (Sklar, 1996), the bivariate cumulative distribution function of the response time and the latent trait can be represented as

$$\mathrm{F}_{T_g,\theta}(t_g,\theta) = \mathrm{C}_g\left(\mathrm{F}_{T_g}(t_g), \mathrm{F}_\theta(\theta)\right), \quad (1)$$

where $\mathrm{C}_g(u_1, u_2)$ is the copula of the cumulative distribution function $\mathrm{F}_{T_g,\theta}(t_g,\theta)$ with domain $[0,1]^2$ and image [0,1]. Note that the transformations $u_g = \mathrm{F}_{T_g}(t_g)$ and $u_\theta = \mathrm{F}_\theta(\theta)$ transform the response time and the work pace into uniformly distributed random variates with support [0,1]. The copula $\mathrm{C}_g(u_1, u_2)$ is the bivariate cumulative distribution function of the transformed random variates U_g and U_θ (Embrechts, Lindskog, & McNeil, 2003).

Latent trait models are often specified via the distribution of the observable indicators when conditioning on the latent trait. In the present case, this is the conditional density function $\mathrm{f}_{T_g|\theta}(t_g \mid \theta)$ of the response time when conditioning on the latent work pace. The conditional density function $\mathrm{f}_{T_g|\theta}(t_g \mid \theta)$ can be represented by the marginal density of the response time $\mathrm{f}_{T_g}(t_g)$ and the conditional density function $\mathrm{f}_{U_g|U_\theta}(u_g \mid u_\theta)$ of the transformed response time U_g when conditioning on the transformed work pace U_θ. As the cumulative distribution function of the two transformed random variates is $\mathrm{C}_g(u_g, u_\theta)$ and U_θ is uniformly distributed in the margin, the conditional density function is just

$$\mathrm{f}_{U_g|U_\theta}(u_g \mid u_\theta) = \mathrm{c}_g(u_g, u_\theta), \quad (2)$$

where $\mathrm{c}_g(u_g, u_\theta) = \partial^2 \mathrm{C}_g(u_g, u_\theta) / \partial u_g \partial u_\theta$ is the copula density function. The item specific conditional density functions (Equation 2) determine the joint distribution of the response times in all items provided that the additional assumption of conditional independence is made.

The conditional independence assumption states that the response times are independent when conditioning on work pace. This should be true in the case that work pace is the only systematic influence on the response times in the different items of the test. Denote by $\mathbf{U_T} = (U_1, \ldots, U_G)$ the transformed response times of a respondent in all G items of the test. Due to the conditional independence, the conditional distribution of the transformed response times $\mathbf{U_T}$ given the transformed work pace U_θ can be factored as

$$\mathrm{f}_{\mathbf{U_T}|U_\theta}(\mathbf{u_T} \mid u_\theta) = \prod_g \mathrm{f}_{U_g|U_\theta}(u_g \mid u_\theta) = \prod_g \mathrm{c}_g(u_g, u_\theta). \tag{3}$$

The marginal density function $\mathrm{f}_{\mathbf{U_T}}(\mathbf{u_T})$ of the transformed response times can be obtained by integrating the joint density of the transformed response times and the transformed work pace $\mathrm{f}_{\mathbf{U_T},U_\theta}(\mathbf{u_T}, u_\theta) = \mathrm{f}_{\mathbf{U_T}|U_\theta}(\mathbf{u_T} \mid u_\theta)\mathrm{f}_{U_\theta}(u_\theta)$ over the transformed work pace. Note that $\mathrm{f}_{U_\theta}(u_\theta)$ can be ignored as U_θ is distributed uniformly over [0,1]. The marginal density function is thus

$$\mathrm{f}_{\mathbf{U_T}}(\mathbf{u_T}) = \int_{[0,1]} \prod_g \mathrm{c}_g(u_g, u_\theta)\, \mathrm{d}u_\theta. \tag{4}$$

The factor copula model requires two specifications: The model for the marginal distribution of the response times $\mathrm{F}_{T_g}(t_g)$ and the copula density function $\mathrm{c}_g(u_g, u_\theta)$ in the different items. Both topics will be addressed in the following section.

Modeling the Marginal Response Time Distribution

The factor copula model requires a model for the marginal response time distribution in the items of the test. As response times in tests are usually skewed, one could use the Weibull distribution, the Gamma distribution or the log-normal distribution, for example. Here, we use the spline hazard model (e.g., Chen, Fan, & Tsyrennikov, 2006; Herndon & Harrell, 1990; Rosenberg, 1986) in the version implemented by Ranger and Kuhn (2015).

The key quantity of the spline hazard model is the item specific hazard function, which is defined as

$$\lambda_{T_g}(t_g) = \lim_{\Delta t \to 0} \frac{\mathrm{P}(t_g \le T_g < t_g + \Delta t \mid T_g \ge t_g)}{\Delta t}. \tag{5}$$

The hazard function $\lambda_{T_g}(t_g)$ is a positive, continuous function with an infinite integral over $[0,\infty)$. It can be interpreted as the tendency to respond in the next moment provided that an individual has not responded before (Klein & Moeschberger, 1997). It thus reflects the instantaneous rate of information processing (Wenger & Gibson, 2004) and taps how individuals acquire information over time. The hazard function is related to the marginal density function of the response time in item g via the relation

$f_{T_g}(t_g) = \lambda_{T_g}(t_g) \cdot \exp\left(-\int_0^{t_g} \lambda_{T_g}(u)\, du \right)$ and to the marginal cumulative distribution function via the relation

$$F_{T_g}(t_g) = 1 - \exp\left(-\int_0^{t_g} \lambda_{T_g}(u)\, du \right). \quad (6)$$

The form of the hazard function determines the shape of the response time distribution. A constant hazard function, for example, implies that the response times have an exponential distribution.

In the spline hazard model, the hazard function is modeled by a spline. Here, we proceed as Ranger and Kuhn (2015) and use M-splines for this purpose (Ramsay, 1988). M-splines are closely related to B-splines but differ with respect to their normalization. With M-splines, the hazard function of an item is modeled piecewisely by a positive polynomial of a certain order. In doing so, the range of the response times is divided into disjoint segments at knots that define the segment borders. In each segment, the hazard function is approximated by a distinct polynomial. The hazard function is then made up of the segment specific polynomials. By imposing constraints on the segment specific polynomials, it can be achieved that the resulting hazard function is positive, continuous and has continuous derivatives up to a certain order. M-splines can be generated by a linear combination of M-spline basis functions. Denote by $M_g^{(k)}(t_g)$ the $k = 1,\ldots,K$ basis functions of the M-spline corresponding to the chosen location of the knots, the number of the segments and the degree of the piecewise polynomial in item *g*. These basis functions are polynomials in some segments and zero otherwise. For lack of space, we go not into details but refer to Ramsay (1988) and an Electronic Supplement for more details. With the M-spline basis, the hazard function in item *g* can be represented by the linear combination

$$\lambda_{T_g}(t_g) = \sum_k \beta_{g[k]} \cdot M_g^{(k)}(t_g), \quad (7)$$

where $M_g^{(k)}(t_g)$ is the k-th basis function of the M-spline basis in item *g* and $\beta_{g[k]}$ its weight. In the spline hazard model, it is common practice to consider the basis functions (and knots) as given and the weights as the model parameters. With the spline representation of the hazard function (Equation 7), the cumulative distribution function in item *g* can easily be recovered by inserting Equation 7 into Equation 6.

There are three different positions concerning the location and the number of knots that define the segment borders. The first position is to consider them as fixed. In this case, the spline hazard model is a parametric model. This position is usually justified by the claim that the spline hazard model provides already a good approximation to the response time distribution with a limited number of knots, as long as the hazard function is not too irregular (smooth and unimodal). Liu and Huang (2008) and Stone and Koo (1986), for example, achieved good results with 5 to 10 knots. The second position is to consider the number of knots as a quantity that increases with the sample size. In case the number of knots increases slower than the sample size, that is, the ratio of both quantities converges to zero, the spline hazard model is a semi-parametric sieve model (Chen et al., 2006; Chen, 2007). In case the number of knots is identical to the sample size (e.g., Whittemore & Keller, 1986), the spline hazard model can be interpreted as a nonparametric model.

Modelling the Association Structure

The copula $\mathrm{C}_g\left(u_g, u_\theta\right)$ and its density function $\mathrm{c}_g\left(u_g, u_\theta\right)$ determine the relation of the response time to the work pace of a respondent; see Equation 2. This poses the question which copula to choose. Numerous parametric variants have been suggested for $\mathrm{C}_g\left(u_g, u_\theta\right)$. Popular parametric copulas are, for example, the bivariate normal copula, the Frank copula or the Gumbel copula; see the book of Joe (2014) or the paper of Michiels and DeSchepper (2008) for an overview. Alternatively, one can use a semiparametric copula in order to avoid strong assumptions about relation of the response time and the work pace (see, e.g., Genest, Ghoudi, & Rivest, 1998; Hernández-Lobato & Suárez, 2011; Lambert, 2007; Michiels & DeSchepper, 2012). Using a semiparametric copula, however, requires very large sample sizes and is not feasible when some variables are latent. In this paper, we will make a compromise between flexibility and manageability. We propose using a normal mixture copula (Nikoloulopoulos & Karlis, 2009; Tewari et al., 2011) with a limited number of mixture components. The normal mixture copula generates the copula by a discrete mixture of bivariate normal distributions. The normal mixture copula subsumes the popular bivariate normal copula as a special case. It can also achieve great flexibility with only a few components (Rossi, 2014, Chapter 1). It is thus capable to account for various relations between the response times and the latent work pace.

The normal mixture copula is generated by mixing multivariate normal distributions. In the simplest case, just two multivariate normal distributions are used, although an extension to more mixture components is straightforward in case more flexibility is required. According to the normal mixture copula, the item specific copula $\mathrm{C}_g\left(u_g, u_\theta\right)$ is a mixture of the cumulative distribution functions of two bivariate normal distributions. Denote by $\Phi_{\mathbf{X}}\left(\mathbf{x}; \boldsymbol{\mu}_g^{(1)}, \boldsymbol{\Sigma}_g^{(1)}\right)$ the cumulative distribution function of the first bivariate normal distribution with expectation $\boldsymbol{\mu}_g^{(1)} = \left(\mu_{g[1]}^{(1)}, \mu_{g[2]}^{(1)}\right)$ and variance covariance matrix $\boldsymbol{\Sigma}_g^{(1)}$ having elements $\boldsymbol{\Sigma}_{g[ij]}^{(1)}$; note that the parameters are considered as item specific. Here, as in the following, the superscript denotes the component and the subscript the item. Specific elements of vectors and matrices are indicated in brackets. The first bivariate normal distribution has the marginal cumulative distribution functions $\Phi_{X_1}\left(x_1;\boldsymbol{\mu}_{g[1]}^{(1)},\boldsymbol{\Sigma}_{g[11]}^{(1)}\right)$ and $\Phi_{X_2}\left(x_2;\boldsymbol{\mu}_{g[2]}^{(1)},\boldsymbol{\Sigma}_{g[22]}^{(1)}\right)$. Denote likewise by $\Phi_{\mathbf{X}}\left(\mathbf{x};\boldsymbol{\mu}_g^{(2)},\boldsymbol{\Sigma}_g^{(2)}\right)$ the cumulative distribution function of the second bivariate normal distribution. This distribution has the marginal cumulative distribution functions $\Phi_{X_1}\left(x_1;\boldsymbol{\mu}_{g[1]}^{(2)},\boldsymbol{\Sigma}_{g[11]}^{(2)}\right)$ and $\Phi_{X_2}\left(x_2;\boldsymbol{\mu}_{g[2]}^{(2)},\boldsymbol{\Sigma}_{g[22]}^{(2)}\right)$. The normal mixture copula is generated by the linear combination

$$\begin{aligned}\mathrm{C}_g\left(u_g,u_\theta\right)=&\ \pi_g^{(1)}\cdot\Phi_{\mathbf{X}}\left(\mathrm{F}_{X_1}^{-1}\left(u_g\right),\mathrm{F}_{X_2}^{-1}\left(u_\theta\right);\boldsymbol{\mu}_g^{(1)},\boldsymbol{\Sigma}_g^{(1)}\right)\\&+\pi_g^{(2)}\cdot\Phi_{\mathbf{X}}\left(\mathrm{F}_{X_1}^{-1}\left(u_g\right),\mathrm{F}_{X_2}^{-1}\left(u_\theta\right);\boldsymbol{\mu}_g^{(2)},\boldsymbol{\Sigma}_g^{(2)}\right),\end{aligned}\tag{8}$$

where $\pi_g^{(1)}$ and $\pi_g^{(2)}$ are the mixing proportions and $\mathrm{F}_{X_1}^{-1}\left(u_g\right)$ and $\mathrm{F}_{X_2}^{-1}\left(u_\theta\right)$ are the inverse functions of the marginal cumulative distribution functions of the normal mixture distribution

$$\mathrm{F}_{X_1}\left(x_1\right)=\pi_g^{(1)}\cdot\Phi_{X_1}\left(x_1;\boldsymbol{\mu}_{g[1]}^{(1)},\boldsymbol{\Sigma}_{g[11]}^{(1)}\right)+\pi_g^{(2)}\cdot\Phi_{X_1}\left(x_1;\boldsymbol{\mu}_{g[1]}^{(2)},\boldsymbol{\Sigma}_{g[11]}^{(2)}\right)\tag{9}$$

$$\mathrm{F}_{X_2}\left(x_2\right)=\pi_g^{(1)}\cdot\Phi_{X_2}\left(x_2;\boldsymbol{\mu}_{g[2]}^{(1)},\boldsymbol{\Sigma}_{g[22]}^{(1)}\right)+\pi_g^{(2)}\cdot\Phi_{X_2}\left(x_2;\boldsymbol{\mu}_{g[2]}^{(2)},\boldsymbol{\Sigma}_{g[22]}^{(2)}\right).$$

The normal mixture copula reduces to the bivariate normal copula by setting one mixing proportion to zero. A factor copula model based on Equation 8 is not identified and some identification restrictions are needed. First, the expectation of the first bivariate normal distribution can be set to zero in all items ($\boldsymbol{\mu}_g^{(1)}=\mathbf{0}$). The variances of the bivariate normal distributions are also not identified and can be set to one ($\boldsymbol{\Sigma}_{g[11]}^{(1)}=\boldsymbol{\Sigma}_{g[22]}^{(1)}=1$, $\boldsymbol{\Sigma}_{g[11]}^{(2)}=\boldsymbol{\Sigma}_{g[22]}^{(2)}=1$). Apart from these identification restrictions, that do not affect the fit of the model, some further restrictions will be imposed. This is due to the need to simplify the model. The model is highly parameterized and thus hard to estimate in smaller samples. We propose setting the two expectation parameters of the second mixture component to the same value ($\boldsymbol{\mu}_{g[1]}^{(2)}=\boldsymbol{\mu}_{g[2]}^{(2)}=\mu_g^{(2)}$). This restriction renders the copula permutation-symmetric $\mathrm{C}_g\left(u_g,u_\theta\right)=\mathrm{C}_g\left(u_\theta,u_g\right)$.

The restricted normal mixture copula is flexible enough to approximate several popular parametric copulas well. Figure 2 contains contour plots (dotted black lines) of the density functions that are generated by the factor copula model (Equation 1) with different copulas. The density functions can be obtained by differentiating the cumulative distribution function with respect to T_g and θ. For the plots, we used the Gumbel copula with parameter of 3, the Frank copula with parameter of 4, the Clayton copula with parameter of 2 and the AMH copula with parameter of 0.5 (Yan, 2007). In all cases, standard normal margins were used. Although response times are usually not standard normally distributed, we used this distribution for illustrative purposes. With standard normal margins, the contour plots can be compared to the contour plot of the bivariate normal distribution. The red lines depict the density functions of the normal mixture copulas that best approximate the parametric copulas in the least squares sense. The approximation is quite good for the Gumbel copula, the Frank copula and the AMH copula. For the Clayton copula, the normal mixture copula does not perform as well. This indicates that the normal mixture copula with two components is flexible, but can not represent all association structures equally well.

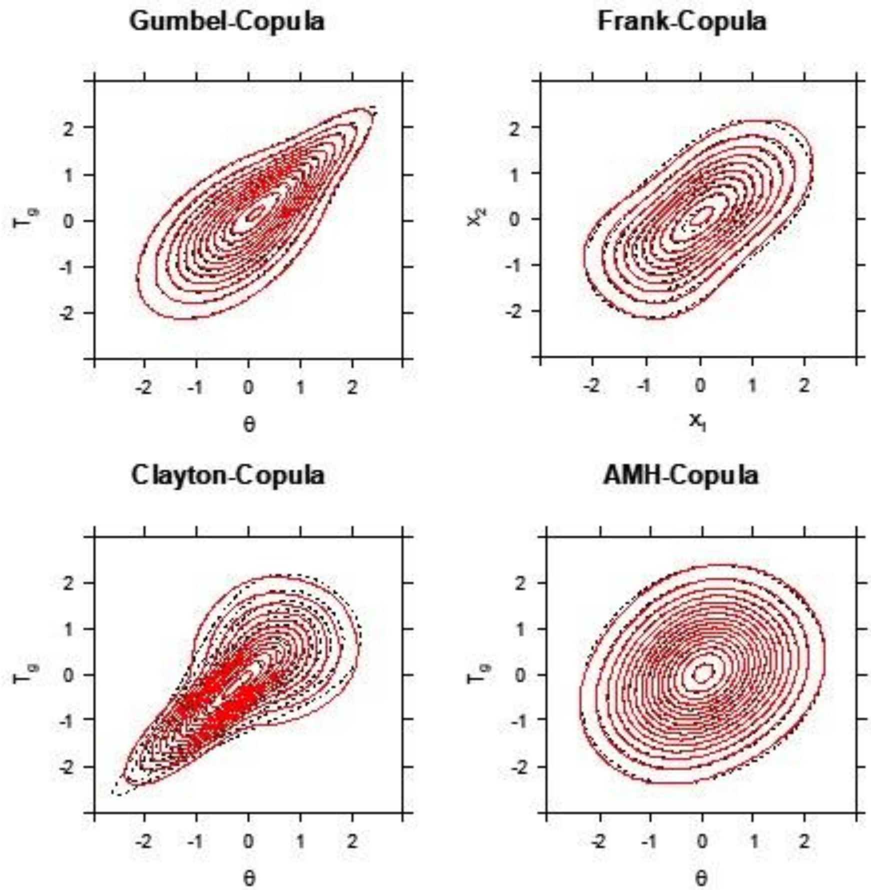

Figure 2

Plots of Several Parametric Copulas and the best Approximating Normal Mixture Copula

Note. Figure 2 visualizes contour plots of the density functions of the response time T_g and work pace θ that are implied by the factor copula model (Equation 1) for different parametric copulas and standard normal marginals (dotted black line) as well as the density functions implied by the normal mixture copulas that best approximate the parametric copulas in the least squares sense (red line).

Summary of the Suggested Model

In the linear transformation model, the marginal distribution of the response times and the relation of the response times to the latent work pace are modeled jointly. This requires that one transformation is capable to transform the response times in such a way that both, the marginal response time distribution and the relation of the response times to the latent work pace, can be represented by a standard latent trait model. This is different in the factor copula model where the marginal response time distributions and the relations of the response times to the latent trait are modeled separately. This has the advantage that assumptions concerning one aspect has no implications for the other aspect. There is thus no need for a transformation that achieves fit with respect to the marginal response time distribution and the structure of the relation. The factor copula model employs a spline hazard model for the marginal response time distribution and a normal mixture copula for the relation of the response times to the latent traits; see Table 1 for a summary.

Table 1

Overview of the Components of the Factor Copula Model

Component 1: Marginal Response Time Model
Role: Model for marginal response time distribution $\mathrm{f}_{T_g}(t_g)$ in item $g = 1,\ldots,$G
Model: $\mathrm{f}_{T_g}(t_g) = \lambda_{T_g}(t_g) \cdot \exp\left(-\int_0^{t_g} \lambda_{T_g}(u)\,\mathrm{d}u\right)$
with: $\lambda_{T_g}(t_g)$: Hazard function (modeled via a positive piecewise polynomial function)
Interpretation: Hazard function as momentary amount of information processing

Component 2: Normal Mixture Copula
Role: Model for the relation of the response times in item g = 1,..,G to the latent work pace **Model**: $\mathrm{F}_{T_g,\theta}(t_g,\theta)=\mathrm{C}_g\left(\mathrm{F}_{T_g}(t_g),\mathrm{F}_\theta(\theta)\right)$ with $\mathrm{F}_{T_g}(t_g)$: Cumulative distribution function of response time in item g $\mathrm{F}_\theta(\theta)$: Cumulative distribution function of latent work pace $\mathrm{C}_g\left(\mathrm{F}_{T_g}(t_g),\mathrm{F}_\theta(\theta)\right)$: Normal mixture copula **Interpretation**: a) In terms of a latent class model with two latent classes b) In terms of the conditional expectation and variance

Parameter Estimation and Inference Aspects

Parameter estimation is difficult for the factor copula model as the model is highly parameterized. We suggest a two-step estimator that facilitates parameter estimation by separately estimating the parameters of the spline hazard model and the parameters of the normal mixture copula.

Parameter Estimation

In the first step, the spline hazard model is fit to the marginal response time distributions in the single items. For this purpose, the time axis has to be divided into disjoint segments at knots that define the segment borders. The number and locations of the knots is at the moment considered as fixed and given. The chosen knots determine the $k=1,\ldots,K$ basis functions $\mathrm{M}_\mathrm{g}^{(\mathrm{k})}(t_g)$ in an item; see the previous section and the electronic supplement for more details. The hazard function in item g can be represented via the linear combination $\lambda_{T_g}(t_g)=\sum_k \beta_{g[k]}\cdot\mathrm{M}_\mathrm{g}^{(\mathrm{k})}(t_g)$ of the item specific basis functions.

The weights $\boldsymbol{\beta}_g=(\beta_{g[1]},\ldots,\beta_{g[k]})$ are estimated by maximizing the log-likelihood function. Denote by $\mathbf{t}_g=(t_{1g},\ldots,t_{ng},\ldots,t_{Ng})$ the response times of the n = *1,...,N* respondents in a pretest sample. Equation 6 and independent sampling implies that the log-likelihood function of the weights is

$$\mathrm{LL}\left(\boldsymbol{\beta}_g;\mathbf{t}_g\right)=\sum_n\left[\log\left(\lambda_{T_g}\left(t_{ng}\right)\right)-\int_0^{t_{ng}}\lambda_{T_g}\left(x\right)\mathrm{d}x\right] \tag{10}$$

$$=\sum_n\left[\log\left(\sum_k\beta_{g[k]}\mathrm{M}_{\mathrm{g}}^{(\mathrm{k})}\left(t_{ng}\right)\right)-\int_0^{t_{ng}}\sum_k\beta_{g[k]}\mathrm{M}_{\mathrm{g}}^{(\mathrm{k})}\left(x\right)\mathrm{d}x\right].$$

The parameter estimates are those values of the weights that maximize the log-likelihood function. As the log-likelihood function is rather simple, any optimization routine can be used for this purpose. Having determined the weights for all items, the hazard function can be estimated via Equation 7. We will denote this estimate as $\hat{\lambda}_{T_g}\left(t_g\right)$. The estimated hazard function is then used in order to transform the response times via the transformation $\hat{u}_{\mathrm{ng}}=1-\exp\left(-\int_0^{t_{ng}}\hat{\lambda}_{T_g}\left(x\right)\mathrm{d}x\right)$. The transformed response times are used in the second step in order to estimate the parameters of the normal mixture copula.

The copula parameters that have to be estimated are the mixing proportions and the parameters of the bivariate normal distributions. For each item g, these are the four parameters $\boldsymbol{\alpha}_g=\left(\pi_g^{(1)},\mu_g^{(2)},\boldsymbol{\Sigma}_{g[12]}^{(1)},\boldsymbol{\Sigma}_{g[12]}^{(2)}\right)$; see the previous section on the normal mixture copula for more details. Parameters $\boldsymbol{\Sigma}_{g[12]}^{(1)}$ and $\boldsymbol{\Sigma}_{g[12]}^{(2)}$ are coefficients of correlations as the variances have been set to one. Denote by $\boldsymbol{\alpha}=\left(\boldsymbol{\alpha}_1',\ldots,\boldsymbol{\alpha}_G'\right)'$ the vector that contains all copula parameters of the G items. The distribution of the transformed response times $u_g=\mathrm{F}_{T_g}\left(t_g\right)$ depends on the copula parameters via the factor copula model given in Equation 4, where the copula density function has to be replaced by the density implied by the normal mixture copula (Equation 8). Using the estimated transformed response times $\hat{\mathbf{u}}=(\hat{u}_{11},\ldots,\hat{u}_{\mathrm{ng}},\ldots,\hat{u}_{\mathrm{NG}})$ of the $n=1,\ldots,N$ respondents in the $g=1,\ldots,G$ items of the pretest as proxies for their true counterpart, the marginal log-likelihood function of the copula parameters is

$$\mathrm{LL}\left(\boldsymbol{\alpha};\mathbf{u}\right)=\sum_n\log\left(\int_{[0,1]}\prod_g\mathrm{c}_g\left(\hat{u}_{ng},u_\theta\right)\mathrm{d}u_\theta\right). \tag{11}$$

The estimates of the copula parameters are those values that maximize the marginal log-likelihood function. The integral in Equation 11 can not be determined in closed form, but has to be approximated numerically, for example, by Gauss-Legendre Quadrature instead. The likelihood function can be maximized by standard techniques like the gradient descent algorithm or the Newton-Raphson algorithm.

Inference Aspects

The large sample properties of the proposed estimator depend on the position concerning the number and location of the knots. We will consider two cases. In the first case, the number and location of the knots are considered as given and fixed. In this case, the proposed estimator of the weights is a standard maximum likelihood estimator on basis of a potentially misspecified response time model. On that condition, the estimator is consistent in the KL-sense and normally distributed in the limit (Gourieroux, Monfort, & Trognon, 1984; White, 1982). The estimator of the copula parameters is a pseudo maximum likelihood estimator as the parameters of the spline hazard model are fixed to previous estimates when the parameters of the copula are estimated (Ackerberg et al., 2012; Gong & Samaniego, 1981). In the context of copulas, this approach is also called the method of inference functions for margins (Joe, 1997). Pseudo maximum likelihood estimates are consistent and normally distributed in the limit. The variance covariance matrix can be obtained as described below. Using a limited number of knots has the disadvantage that for most response time distributions, the response time model is misspecified. However, as long as the true hazard function is not too irregular, the misspecification is negligible in practice. Simulation studies, for example, suggest that the spline hazard model often performs better than nonparametric models with respect to the root mean-squared error (Rosenberg, 1995). Besides, as models are simplifications of reality, it is unrealistic to assume that a model can be specified correctly. In the second case, the number of knots increases with the sample size, but at a lower rate. In this case, the estimator of the weights is a sieve maximum likelihood estimator (Chen et al., 2006; Chen, 2007). Sieve maximum likelihood estimators are consistent and asymptotic normal under weak regularity conditions. The estimator of the copula parameters is also consistent and normally distributed. Ackerberg, Chen, and Hahn (2012) proved that with respect to parameter inference, the sieve maximum likelihood estimator can be treated as a pseudo likelihood estimator with a large number of parameters in the first step. This simplifies the analysis as irrespective whether the knots are limited and fixed or increase with the sample size, the same approach to parameter inference can be used.

The asymptotic variance covariance matrix of the estimates can be obtained as follows; see Joe (1997) or Ackerberg et al. (2012) for more details. Denote by $\mathbf{s}_{T_g}\left(t_g;\boldsymbol{\beta}_g\right)=\partial\log\left(\mathrm{f}_{T_g}\left(t_g;\boldsymbol{\beta}_g\right)\right)/\partial\boldsymbol{\beta}_g$ the score function of the spline hazard model in item g. The score function contains the first derivatives of the log-transformed marginal response time distribution given in Equation 6 with respect to the weights; note that in contrast to Equation 6, we mention the item parameters explicitly. The score function in all items $\mathbf{s}_{\mathbf{T}}(\mathbf{t};\boldsymbol{\beta})=\left(\mathbf{s}_{T_1}\left(t_1;\boldsymbol{\beta}_1\right)',\ldots,\mathbf{s}_{T_G}\left(t_G;\boldsymbol{\beta}_G\right)'\right)'$ results after stacking the item specific score vectors. Denote by $\mathbf{s}_{\mathbf{U}(\mathbf{T})}(\mathbf{u}(\mathbf{t});\boldsymbol{\alpha},\boldsymbol{\beta})=\partial\log\left(\mathrm{f}_{\mathbf{U}_\mathbf{T}}\left(\mathbf{u}_\mathbf{T};\boldsymbol{\alpha},\boldsymbol{\beta}\right)\right)/\partial\boldsymbol{\alpha}$ the score function of the factor copula model. The score function contains the derivatives of the density function in Equation 4 with respect to the copula parameters; note that the density function depends on the copula parameters via Equation 8 and on the spline hazard model due to the transformation $u_g = \mathrm{F}_{T_g}(t_g)$. The combination of both score functions $\mathbf{s_T}(\mathbf{T};\boldsymbol{\beta})$ and $s_{\mathbf{U_T}}(\mathbf{U_T};\boldsymbol{\alpha},\boldsymbol{\beta})$ in one vector is denoted as $\mathbf{s_T}(\mathbf{T};\boldsymbol{\gamma})$. Here, $\boldsymbol{\gamma} = (\boldsymbol{\beta}',\boldsymbol{\alpha}')'$ represents all parameters of the factor copula model, that is, the weights of the spline hazard model and the parameters of the normal mixture copulas. The asymptotic variance covariance matrix of the two-step estimator can be determined as

$$\boldsymbol{\Sigma}_\gamma = 1/n \cdot \left[\mathrm{E}\left(\partial\mathbf{s}(\mathbf{T};\boldsymbol{\gamma})/\partial\boldsymbol{\gamma}'\right)\right]^{-1} \mathrm{E}\left[\mathbf{s}(\mathbf{T};\boldsymbol{\gamma})\mathbf{s}(\mathbf{T};\boldsymbol{\gamma})'\right]\left[\mathrm{E}\left(\partial\mathbf{s}(\mathbf{T};\boldsymbol{\gamma})/\partial\boldsymbol{\gamma}'\right)\right]^{-1}, \qquad (12)$$

where $\partial\mathbf{s}(\mathbf{T};\boldsymbol{\gamma})/\partial\boldsymbol{\gamma}'$ is the Hessian matrix. The variance covariance matrix has the typical sandwich structure that arises in pseudo maximum likelihood estimation. In practice, the values of the item parameters are replaced by their estimates. Expectations are approximated by sample means.

Although the factor copula model is very flexible, it can provide a poor representation of the data in practice. It is therefore advisable to evaluate the global fit of the model. Although several tests of the global fit of copula models exist – see the overviews given in Berg (2009), Fermanian (2013) and Genest, Rémillard, and Beaudoin. (2009) – their application is not feasible in the present context, as the tests are extremely computer intensive. Besides, as all models are necessarily wrong, it usually is more informative to assess whether a model provides a good representation of the data than to test whether it is true. Graphical checks are the preferred tool for an evaluation of approximate model fit. In the context of copula models, a crucial question concerns the model's capability to account for the association of the data. This can be checked by a multivariate generalization of the P-P-plots. For the plot, the response times have to be transformed according to the Rosenblatt transformation (Dobrić & Schmid, 2007; Rosenblatt, 1952). The Rosenblatt transformation is a multivariate generalization of the univariate probability integral transformation $U = \mathrm{F}(X)$ and transforms the dependent response times into independent and uniformly distributed random variates with support of [0,1] in case the model is valid.

Some word of caution is needed with respect to model comparison. It is tempting to test for the need of the second mixture component by testing whether the second mixture proportions deviate from zero ($\boldsymbol{\pi}_2^{(g)} = \mathbf{0}$). This is a hypothesis on the boundary of the parameter space. The standard asymptotic results for maximum likelihood estimates do not apply in this case (Self & Liang, 1987). The model is also not identified under the null hypothesis (Lo, Mendell, & Rubin, 2001). In order to evaluate the number of mixture components or to choose the most adequate copula, it is better to compare alternative models with respect to another information criterion (AIC) of Akaike (1992) or the Bayes information criterion (BIC) of Schwarz (1978). The values of the information criteria are comparable over different copulas as long as the same transformation of the response times is used. Although the application of information criteria is controversial when parameters are on the boundary of the parameter space

(Hughes & King, 2003) these criteria usually work well in practice (Jordanger & Tjostheim, 2014).

Simulation Study

In order to assess the performance of the two-step estimator, a simulation study was conducted. For the study, the response times in a test of 12 items were simulated. The data were generated as follows. For each fictitious respondent, a level of work pace u_θ was drawn from the uniform distribution with range [0,1]. Then, the transformed response times u_g were generated by random draws from the conditional distribution $\mathrm{f}_{U_g|U_\theta}\left(u_g \mid u_\theta\right)$ given in Equation 2 that was implied by the normal mixture copula given in Equation 8. The parameters of the normal mixture copula were chosen such that the resulting copula resembled Gumbel copulas with copula parameters of 3.0, 2.0 and 1.5. In items 1–4, the parameters were $\pi_g^{(1)} = 0.30$, $\mu_g^{(2)} = -1.20$, $\boldsymbol{\Sigma}_{g[12]}^{(1)} = 0.95$ and $\boldsymbol{\Sigma}_{g[12]}^{(2)} = 0.75$. In items 5–8, the parameters were $\pi_g^{(1)} = 0.30$, $\mu_g^{(2)} = -1.00$, $\boldsymbol{\Sigma}_{g[12]}^{(1)} = 0.90$ and $\boldsymbol{\Sigma}_{g[12]}^{(2)} = 0.50$. In items 9–12, the parameters were $\pi_g^{(1)} = 0.50$, $\mu_g^{(2)} = -0.60$, $\boldsymbol{\Sigma}_{g[12]}^{(1)} = 0.75$ and $\boldsymbol{\Sigma}_{g[12]}^{(2)} = 0.15$. The transformed response times u_g were finally transformed back into the corresponding quantiles of a Weibull distribution. Three different Weibull distributions were considered. We used a Weibull distribution with shape parameter of 2 and scale parameter of 2 in item 1–4, a Weibull distribution with shape parameter of 1 and scale parameter of 2 in item 5–8, and a Weibull distribution with shape parameter of 2 and scale parameter of 1 in item 9–12. Two sample sizes were considered, namely samples of 1000 respondents and samples of 5000 respondents. For each of the two simulation conditions, 250 simulation samples were generated. All calculations were performed within the statistical environment R (R Development Core Team, 2009). The scripts can be obtained from the corresponding author on request.

The simulation data sets were analyzed as described in the previous section. In a first step, the spline hazard model was fit to the marginal distributions of the response times by maximizing the log-likelihood function in Equation 10. The spline hazard model was implemented with four knots located approximately at the terciles of the response time distributions. The cumulative distribution functions of the fitted spline hazard model were used in order to transform the response times according to Equation 6. The factor copula model was then fitted to the transformed response times by maximizing the likelihood function in Equation 11. For maximization, we first used the steepest ascend algorithm. After 30 steps, we switched to the Newton-Raphson algorithm for sake of speed. The integrals over u_θ in the marginal log-likelihood function

(Equation 11) and its first and second derivatives were approximated via Gauss-Legendre Quadrature using 100 nodes. Having estimated the model parameters, Wald-type confidence intervals were calculated for a confidence level of 0.95. The required standard errors of estimation were determined with Equation 12. Expectations were replaced by sample averages.

The results of the simulation study concerning the recovery of the copula parameters are reported in Table 2. There, the true values of the copula parameters are given, as well as the average estimate, the standard error of estimation and the coverage frequency of the confidence intervals; note that the coverage frequency should be close to 0.95. Results for copula parameters with the same value have been averaged over the items. Further results, for example, about the recovery of the hazard functions, can be obtained from the corresponding author.

Table 2

Parameter Recovery of the Copula Parameters for Two Samples Sizes as well as the Performance of Wald-type Confidence Intervals in the Simulated Data Sets

N = 1000												
Par	$\pi_g^{(1)}$			$\mu_g^{(2)}$			$\boldsymbol{\Sigma}_{g[12]}^{(1)}$			$\boldsymbol{\Sigma}_{g[12]}^{(2)}$		
Item	1–4	5–8	9–12	1–4	5–8	9–12	1–4	5–8	9–12	1–4	5–8	9–12
TV	0.30	0.30	0.50	-1.20	-1.00	-0.60	0.95	0.90	0.75	0.75	0.50	0.15
M	0.29	0.30	0.49	-1.26	-1.04	-0.64	0.95	0.90	0.75	0.75	0.50	0.14
SE	0.09	0.08	0.13	0.28	0.23	0.20	0.02	0.03	0.06	0.03	0.05	0.13
p	0.94	0.94	0.92	0.94	0.94	0.95	0.96	0.95	0.93	0.95	0.94	0.93
N = 5000												
Par	$\pi_g^{(1)}$			$\mu_g^{(2)}$			$\boldsymbol{\Sigma}_{g[12]}^{(1)}$			$\boldsymbol{\Sigma}_{g[12]}^{(2)}$		
Item	1–4	5–8	9–12	1–4	5–8	9–12	1–4	5–8	9–12	1–4	5–8	9–12
TV	0.30	0.30	0.50	-1.20	-1.00	-0.60	0.95	0.90	0.75	0.75	0.50	0.15
M	0.30	0.30	0.50	-1.21	-1.01	-0.60	0.95	0.90	0.75	0.75	0.50	0.15
SE	0.04	0.03	0.05	0.11	0.09	0.08	0.01	0.01	0.02	0.01	0.02	0.05
p	0.95	0.95	0.95	0.96	0.95	0.95	0.94	0.95	0.95	0.94	0.95	0.95

Note. Results are based on 250 simulation samples per condition. Results have been averaged over items with the same parameter value. Wald-type confidence intervals were determined for $\alpha = 0.05$. TV: true value; *M*: average estimate; *SE*: standard error of estimation; *p*: relative coverage frequency of confidence intervals.

Table 2 demonstrates that the estimation procedure performs well in samples of moderate size. In samples of 1000 respondents, the estimates are almost unbiased. The coverage frequency of the Wald-type confidence intervals is slightly too low in the mixing proportion $\pi_g^{(1)}$ and the copula parameters $\boldsymbol{\Sigma}_{g[12]}^{(1)}$ and $\boldsymbol{\Sigma}_{g[12]}^{(2)}$. In samples of 5000 respondents, the bias disappears and the coverage frequencies are close to the intended ones.

Estimation of Latent Traits

The latent trait model can be used as a measurement model in order to infer the latent traits of the respondents. In doing so, we consider the parameters of the factor copula model as known. This is standard practice in psychological assessment in case the item parameters have been estimated with a large pretest sample. The latent traits can be estimated with maximum likelihood estimation or Bayes estimation. In the following, we describe the maximum likelihood estimator in more detail. Basis of the estimation are the transformed response times $\mathbf{u_t}$ of a respondent in the items of the test; see Equation 6. The conditional distribution of the transformed response times given the transformed work pace u_θ is just the product of the copula density functions; see Equation 3. Contrary to the previous sections, the work pace is now interpreted as a parameter that determines the distribution of the transformed response times, not as a random variate. The work pace is estimated by maximizing the log-likelihood function

$$\log\left(\mathrm{f}_{\mathbf{U_T}|U_\theta}\left(\mathbf{u_t} \mid u_\theta\right)\right) = \sum_g \log\left(c_g\left(u_g, u_\theta\right)\right) \tag{13}$$

over u_θ. As this estimator is a standard maximum likelihood estimator, the large sample theory of maximum likelihood estimation applies. That is, the distribution of the estimator converges to the normal distribution with expected value of u_θ. The variance of the estimator can be derived with the information matrix, that is, the expectation of the second derivation of the log-likelihood function with respect to u_θ.

Simulation Study

In order to assess the applicability of the factor copula model in psychological assessment, a second simulation study was conducted. The second simulation study was based on the results from the first simulation study, as the fitted factor copula models were used as measurement models for new observations. In doing so, we mimicked the standard approach in psychological assessment as closely as possible. The proceeding was as follows. For each of the 250 pretest samples of the first simulation study, a set of additional response time patterns was simulated with the factor copula model and the true parameter values. The new response time patterns were generated for 10 levels of u_θ that were equally spaced from 0.05 to 0.95. This corresponds to work pace levels from $\theta = -1.64$ to $\theta = 1.64$. For each of the 10 levels of u_θ, we generated 20 response time patterns $\mathbf{u_t}$. For each of the 10×20 response time patterns, the maximum likelihood estimator of u_θ was determined. The unknown parameters of the factor copula model were replaced by the marginal maximum likelihood estimates of the corresponding pretest sample. In this way, altogether 20×10×250 response time patterns were analyzed. Having estimated the transformed work pace u_θ, the observed information matrix was determined and the standard error of estimation was derived. The standard error of estimation was then used for the calculation of Wald-type confidence intervals with confidence level of $c = 0.95$. The results of the second simulation study are reported in Table 3. There, the true trait level, the average estimate, the standard error of estimation and the coverage frequency of the confidence intervals are reported.

Table 3

Performance of the Maximum Likelihood Estimator of the Transformed Work Pace u_θ When Using the Factor Copula Model as a Measurement Model for Two Sizes of the Pretest Sample

					N = 1000					
u_θ	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
M	0.06	0.16	0.26	0.35	0.45	0.54	0.64	0.74	0.84	0.95
SE	0.04	0.08	0.10	0.10	0.10	0.10	0.08	0.07	0.04	0.02
p	0.91	0.91	0.91	0.91	0.90	0.91	0.92	0.93	0.92	0.91
					N = 5000					
u_θ	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
M	0.06	0.16	0.26	0.35	0.45	0.54	0.65	0.74	0.84	0.95
SE	0.04	0.08	0.09	0.10	0.10	0.09	0.08	0.06	0.04	0.02
p	0.91	0.91	0.92	0.92	0.92	0.92	0.93	0.93	0.93	0.94

Note. Results are based on 20 × 250 response time patterns per trait level. Wald-type confidence intervals were determined for $\alpha = 0.05$. *N*: Size of pretest sample; TV: true value; *M*: average estimate; *SE*: standard error of estimation; *p*: relative coverage frequency of confidence intervals.

The estimates are virtually unbiased. The coverage frequencies of the confidence intervals, however, are too low. As the undercoverage does not depend on the size of the pretest sample, it can not be the result of the additional variation caused by the estimation of the model parameters. The undercoverage is probably due to the small number of observations. As the estimates are based on only 12 response times, the large sample theory of maximum likelihood estimation is not applicable.

Empirical Application

In order to demonstrate the usefulness of the model, a real data set was analyzed. The data set consisted of the response times in a test of choice reactions (Kuhn, Raddatz, Holling, & Dobel, 2013). The test was computer-assisted and served in order to assess processing speed without numerical or verbal content. Participants were presented with a white square on an initially black screen, and they quickly had to decide whether the square was either left or right of a white vertical line at the center of the screen by pressing one of two response keys (F or J on a standard keyboard). After 5 instruction items with feedback, the test consisted of 20 items with a total time limit of 1 minute. Overall, 1207 elementary school children (grades 2 to 4) participated. Mean age was 107.73 months (SD = 10.46). The study was carried out in accordance with the ethical guidelines of the German Society of Psychology (DGPs). The protocol was approved by the local ethics committee of the university of Münster. The parents of all children gave written informed consent in accordance with the Declaration of Helsinki.

First, the data were screened for outliers. Response times are usually very messy and a minority of respondents with low motivation can distort the results tremendously. We considered response times as outlying when they deviated from the median for more than four times the interquartile range. All participants with outlying response times were removed from the data set. This reduced the sample size from 1207 to 1140 respondents. Then, the spline hazard model was fit to the response time distributions in the single items. A first model with two interior knots could not represent the response time distributions well. We therefore increased the number of knots. A model with 8 interior knots had an excellent fit according to a graphical check of model fit with P-P-plots.

The estimated cumulative hazard functions did not resemble the cumulative hazard functions of standard response time distributions. In most items, the cumulative hazard function was near zero for some initial period of time. This indicates that the response time distributions are shifted. In order to improve the fit of the response time model further, we subtracted the shift from the response times. As the shift we used the time point where the cumulative hazard function started to deviate from zero. The spline hazard model was then fitted to the shifted response times again. The fitted spline hazard models were then used in order to transform the response times according to Equation 6. The factor copula model with the normal mixture copula was fit to the transformed response times. The integral over work pace (see Equation 4) was approximated by Gauss-Legendre Quadrature and 100 nodes. The marginal likelihood function was maximized as described in the simulation section. In addition to the proposed model, we considered two alternative versions of the factor copula model. The alternative versions were based on different copulas, namely the Gumbel copula and the standard normal copula; note that the model with the standard normal copula is similar to a linear transformation model with normally distributed residuals. Both alternative models were fit to the data via marginal maximum likelihood estimation similarly to the normal mixture copula model.

Of the three competing factor copula models, the model with the normal mixture copula had the best model fit according to the information criterion of Akaike (AIC). The version with the normal mixture copula had an AIC of -11225.74, whereas the version with the Gumbel copula had an AIC of -10697.73 and the version with the normal copula had an AIC of -10958.70. When determining the AIC values, the effect of transforming the response times to uniformly distributed random variates (see Equation 6) was ignored. In addition to the relative model fit, we evaluated the global fit of the factor copula model with the normal mixture copula. For this purpose, we compared the empirical and theoretical conditional distribution of the response times when conditioning on the estimated latent trait values of the participants graphically. This proceeding is similar to residual analysis in standard factor models. None of the comparisons gave reasons for concern. This implies that the factor copula model is capable to represent the data.

The factor copula model can give insight into the response process of the respondents. The hazard function, for example, reflects the momentary rate of information processing. Hazard functions with a sharp peak indicate an automated response mode that is governed by insight, whereas a more constant hazard function implies more gradual problem processing. This can be used, for example, in simple numerical tasks (addition/subtraction problems) in order to get a better understanding on how a problem is approached. The normal mixture copula model, on the other hand, can be interpreted as a latent class model. Mixing proportions that are similar in different items indicate the existence of latent classes of respondents with different response modes. Different mixing proportions, on the other hand, exclude the existence of different response modes. As the items of the test were simple decision tasks, where different response strategies are improbable, we did not undertake such interpretations. We focused on the usage of the model as a measurement model instead. In doing so, we regarded the normal mixture copula simply as a flexible density estimator. We evaluated the relation of the response times and the latent trait that is implied by the normal mixture copula.

The estimates and standard errors of estimation of the parameters of the normal mixture copula are reported in Table 4; note that the mixing proportions in several items deviate from zero and one. This corroborates that the neither the popular log-normal factor model nor the linear transformation model is capable to represent the association of the response times well.

Table 4

Estimates (Standard Errors of Estimation) for the Parameters of the Normal Mixture Copula in the 20 Items of the Test Assessing Processing Speed

Item	$\pi_g^{(1)}$	$\mu_g^{(2)}$	$\mathbf{\Sigma}_{g[12]}^{(1)}$	$\mathbf{\Sigma}_{g[12]}^{(2)}$
1	0.52 (0.19)	-0.46 (0.16)	0.25 (0.11)	0.73 (0.07)
2	0.85 (0.12)	-0.43 (0.19)	0.58 (0.04)	0.78 (0.08)
3	1.00 (0.00)	-3.90 (0.37)	0.69 (0.02)	1.00 (0.00)
4	0.06 (0.04)	-1.32 (0.33)	0.90 (0.08)	0.67 (0.03)
5	0.28 (0.14)	-0.00 (0.00)	0.44 (0.13)	0.82 (0.03)
6	0.97 (0.01)	-2.43 (0.45)	0.79 (0.02)	1.00 (0.00)
7	0.03 (0.01)	-2.83 (0.29)	-0.27 (0.34)	0.75 (0.02)
8	0.34 (0.22)	-0.66 (0.43)	0.91 (0.04)	0.55 (0.08)
9	0.04 (0.02)	-0.00 (0.00)	0.00 (0.19)	0.80 (0.01)
10	0.46 (0.16)	-0.00 (0.00)	0.54 (0.09)	0.89 (0.04)
11	0.14 (0.06)	-0.69 (0.28)	0.94 (0.04)	0.67 (0.03)
12	0.63 (0.16)	-0.60 (0.22)	0.77 (0.05)	0.31 (0.14)
13	0.41 (0.24)	-0.43 (0.19)	0.43 (0.16)	0.85 (0.06)
14	0.55 (0.13)	-0.00 (0.00)	0.80 (0.03)	0.49 (0.07)
15	0.33 (0.37)	-0.00 (0.00)	0.54 (0.20)	0.82 (0.07)

16	0.95 (0.04)	-1.60 (0.61)	0.63 (0.03)	0.92 (0.05)
17	0.29 (0.12)	-0.00 (0.00)	0.35 (0.13)	0.82 (0.03)
18	0.31 (0.15)	-0.00 (0.00)	0.34 (0.15)	0.80 (0.04)
19	0.54 (0.16)	-0.98 (0.31)	0.70 (0.09)	0.25 (0.17)
20	0.36 (0.19)	-0.49 (0.18)	0.30 (0.15)	0.76 (0.05)

In order to give an impression of the form of the resulting copulas, the underlying mixture normal distributions in four exemplary items are plotted in Figure 3. Figure 3 illustrates the difference to the linear transformation model where the transformed response times and the latent trait are assumed to follow a bivariate normal distribution. The estimated copulas deviate sometimes significantly from the elliptical contours implied by the bivariate normal distribution.

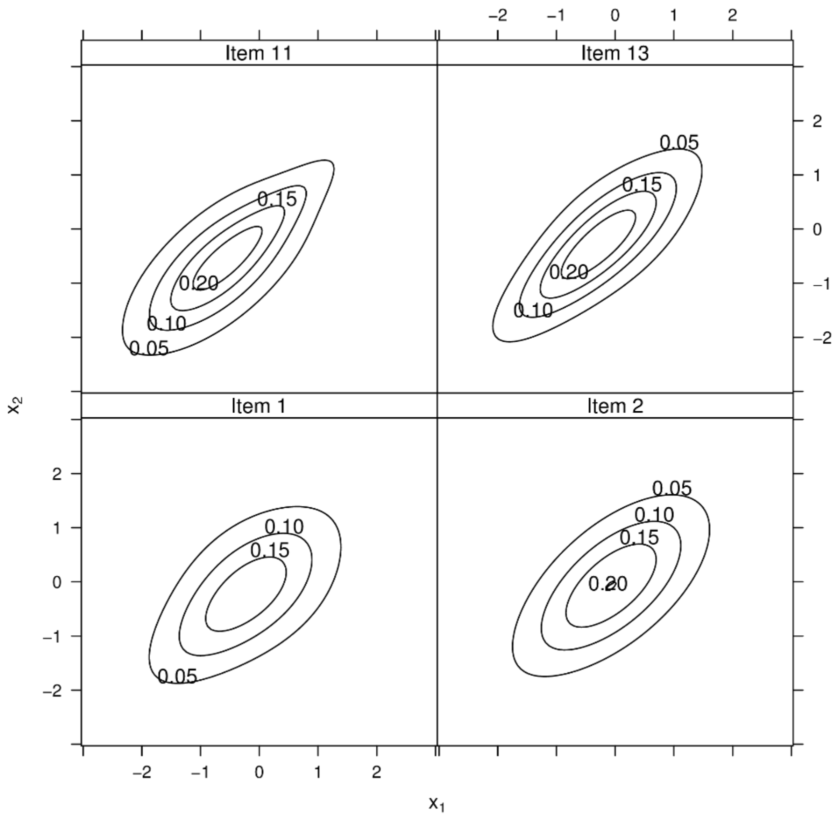

Figure 3

Contour Plots of the Mixture Normal Distribution in Four Items of the Test

Note. Contour plot of the mixture normal distribution that generates the copula (Equation 8 / Equation 9) in four items of the test for processing speed. The axes x_1 and x_2 denote the arguments of the mixture normal distribution. In terms of the factor copula model, these are the values of the transformed response times $x_1 = F_{X_1}^{-1}\left(F_{T_g}\left(t_g\right)\right)$ and transformed work pace $x_2 = F_{X_2}^{-1}\left(F_\theta\left(\theta\right)\right)$; see Equation 6 and Equation 9. The contour lines can directly be compared to the elliptic contours of a bivariate normal distribution.

In order to investigate the implications of the factor copula model, we analyzed the relation between the work pace and the response times in more detail. In specific, we determined the conditional expectation and the conditional standard deviation of the log-transformed response times that are implied by the factor copula model for different levels of the work pace. We log-transformed the response times as the most popular response time model in psychological assessment, the log-normal factor model of van der Linden (2006), claims that the log-response times are related to the work pace according to a linear and variance homogeneous regression model. The true relation between the implied conditional moments and the latent work pace is visualized in Figure 4 for four exemplary items. The expected log-response time is almost a linear function of the work pace. This conforms to the predictions from the log-normal factor model. The standard deviation, however, is not constant, but increases with more extreme levels of work pace. In this aspect, our model differs from the log-normal factor model. This is an important finding as the conditional standard deviation has a critical role in trait inference; see the illustration in the first section of the paper. The finding also indicates that the response process might be unstable in extreme levels of work pace.

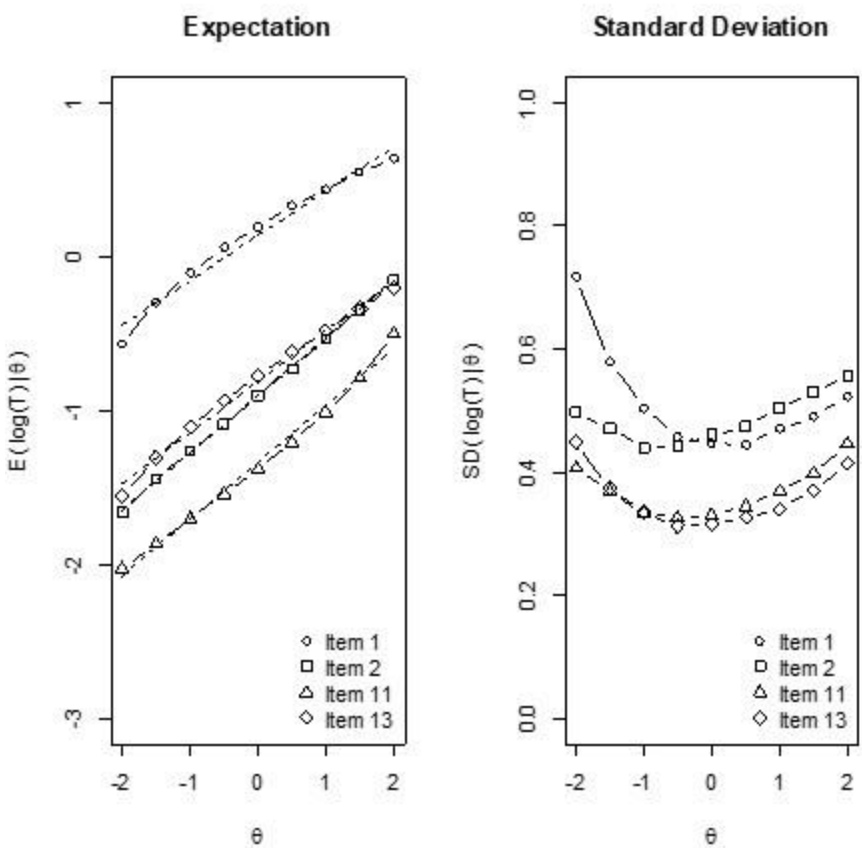

Figure 4

Conditional Expectation and Standard Deviations of the Log-Response Times in Four Items

Note. Conditional expectations and standard deviations of the log-response times when conditioning on work pace as implied by the factor copula model with a normal mixture copula in four items of the test for processing speed.

Discussion

Response time modeling has become popular in psychological assessment. The choice of a response time model, however, is not easy as long as there is no strong theory about the response process that guides the selection. Flexible response time models have the advantage that one does not have to commit oneself to a narrow class of response time distributions or association structures. In this paper, we suggest modeling the response times in tests with the factor copula model and the normal mixture copula. The proposed model is novel in the sense that the combination of several known elements is new and the factor copula model has never been used for response times in tests before. In contrast to previous response time models, the factor copula model has a decisive advantage. In the factor copula model, the marginal distribution and the association structure are decoupled. It is thus possible to model both components separately. The spline hazard model that is used for the marginal response time distribution is simple, but flexible and performs well in practice. The normal mixture copula subsumes the normal copula and is capable to approximate popular Archimedean copulas well. This is important as the log-normal factor model (van der Linden, 2006) and its generalization to the linear transformation model (Ranger & Kuhn, 2013) imply a normal copula and the popular frailty model an Archimedean copula (Marshall & Olkin, 1988; Oakes 1989). The suggested model can thus be seen an extension of existing response time models. This, however, does not mean that the proposed model can represent any kind of multivariate response time data. The normal mixture copula with two mixture components might not be flexible enough to approximate any copula like, for example, copulas with asymmetric tail dependence. In this case, more mixture components are needed (Rossi, 2014, Chapter 1). The proposed model is also one-dimensional, that is, considers just one latent trait. This conforms to standard practice in response time modeling (van der Linden, 2011; Stadler, Radkowitsch, Schmidmaier, Fischer, & Fischer, 2020). Psychological tests, however, often consist of several subscales that assess different aspects of a construct. In this case, one has to use subscale specific response time models (Zhan, Jiao, Kaiwen, Wang, & Keren, 2021). The proposed response time model can easily be extended to this. A multidimensional response time model with compensatory latent traits, however, is hard to implement. Although in theory, the normal mixture copula could be extended to more than one latent trait by mixing multivariate normal distributions, this soon becomes numerically intractable. The model is thus not ideal for an exploratory analysis of the trait structure. In personality scales, the response times are sometimes not only related to the work pace of a respondent, but also to the latent trait underlying the responses as respondents with a very low or a very high level of the trait tend to respond faster (Ferrando, 2006). Modeling this effect again requires a multidimensional copula. Model fitting, however, might be facilitated by incorporating information from the responses. This, however, is a topic of future research.

The combination of the spline hazard model and the factor copula model with the normal mixture copula was not entirely motivated by the flexibility of the model, but also by its interpretability. The hazard function that is estimated in the first step can be interpreted as the average rate of information acquisition (Wenger & Gibson, 2004). This allows the comparison of information acquisition in different items. It might also be possible to relate features of the items to features of the hazard function. The normal mixture copula can additionally be motivated by the fact that response times distributions in tests often result from mixtures of several processes. Responses for example can originate from automatic versus controlled processes (Böckenholt, 2012; Kahneman, 2011). In this case, the mixture model is not simply a flexible density estimator, but represents real subgroups.

The actual model is for the response times only. It mainly makes the relation between the latent trait and the response times in single items more flexible. A more general version could also relax the conditional independence assumption by assuming dependencies between response times in different items. One could also combine the factor copula model for the response times with an item response model according to the hierarchical model framework of van der Linden (2011). In this model, one could relax the conditional independence assumption concerning the responses and response times with the copula approach. These topics all deserve future research.

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